A cylindrically symmetric solution in Einstein-Maxwell-dilaton gravity[4]

P. Klepáč [1 2] and J. Horský [1 3]

We consider the existence of Einstein-Maxwell-dilaton plus fluid system for the case of stationary cylindrically symmetric spacetimes. An exact inhomogeneous ε-order solution is found, where the parameter ε parametrizes the non-minimally coupled electromagnetic field. Some its physical attributes are investigated and a connection with already known Gödel-type solution is given. It is shown that the found solution also survives in the string-inspired charged gravity framework. We find that a magnetic field has positive influence on the chronology violation unlike the dilaton influence.

KEY WORDS: exact solutions, charged perfect fluid, scalar field

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[1]Institute of Theoretical Physics and Astrophysics, Faculty of Science, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic
[2]E-mail: klepac@physics.muni.cz
[3]E-mail: horsky@physics.muni.cz

1 Introduction

Einstein's theory of relativity is, in general, an excellent approximation of gravitational phenomena which appear at low energies. Nevertheless, if one goes to energies at the Planck scale, then one is faced with the neccessity of introducing quantum corrections. Nowadays, superstring theory is believed to unify succesfully all of the known fundamental interactions observed in nature. Moreover the original Einstein's theory naturally emerges if one ignores all higher-order stringy corrections.

In the last decade the string cosmology has become an attractive subject of interest. As one goes to the low energies, string cosmology is actually the classical cosmology with general relativity enriched by addition of massless scalar fields. An incorporation of these fields is perhaps promissible way how to resolve long standing problems in cosmology.

A number of solutions to the so called Einstein-Maxwell-dilaton gravity, i.e. relativistic gravity theory containing non-minimally coupled dilaton and electromagnetic fields, has been derived by various techniques [1]. Barrow and Dąbrowski [2] obtained stringy Gödel-type solution without closed timelike curves (CTC's) by considering the one-loop corrected superstring effective action. Kanti and Vayonakis [3] have extended analysis of [2] on case with an electromagnetic field, and their found that the chronology is violated. Also results of others authors show that presence of the electromagnetic field may cause the chronology violation (especially purely magnetic filed parallel to the rotation axis) while a scalar field may again restore the chronology [4, 5].

In this work some results on the stationary cylindrically symmetric spacetimes in Einstein-Maxwell-dilaton (EMD) theory of gravity are presented. The reason for studying this class of the spacetimes is twofold. First, searching for EMD solutions is important in itself. Second, in the classical relativity theory the cylindrically symmetric spacetimes are known to violate some of the chronology conditions [6]. Therefore it is natural to address the question of chronology violation in the EMD spacetimes. The paper extends the results of [2, 3] to the inhomogeneous case where, in general, only three isometries are present.

The paper is organized as follows. After some preliminaries in section 2 we derive in section 3 the exact solution for the lowest order in parameter ε, which parametrizes the electromagnetic field. In section 4 there are the results of the previous one generalized on the ε-order corrections in the framework of the EMD theory. The case with more scalar fields is studied in section 5 provided that in zero-order they depend solely on the longitudinal direction. In section 6 it is shown that the found solution in fact still applies even if string-inspired charged gravity is taken under consideration. Finally section 7 briefly summarizes the basic properties of this solution.

2 Preliminaries

We search for cylindrically symmetric stationary spacetimes. Then there exist local coordinate systems $(x^0, x^1, x^2, x^3) = (t, \varphi, z, r)$ adapted to Killing fields $\partial_t,\ \partial_\varphi,\ \partial_z$, where the hypersurfaces $\varphi = 0$ and $\varphi = 2\pi$ are to be identified and ∂_t is everywhere a nonvanishing timelike field.

Furthermore we choose a local coframe fields $\Theta^{\hat{\mu}}$ defined by (Greek indices run from 0 to 3)

$$\begin{aligned} \Theta^{\hat{0}} &= \mathrm{e}^{\alpha}(\,\mathrm{d}t + f\,\mathrm{d}\varphi)\ , \qquad & \Theta^{\hat{1}} &= l\,\mathrm{d}\varphi\ , \\ \Theta^{\hat{2}} &= \,\mathrm{d}z\ , & \Theta^{\hat{3}} &= \mathrm{e}^{\delta}\,\mathrm{d}r\ , \end{aligned} \tag{2.1}$$

with $f,\ l,\ \alpha,\ \delta$ being functions of r only.

Let the metric tensor field be in the basis (2.1) written as $g = \eta_{\mu\nu}\Theta^{\hat{\mu}} \otimes \Theta^{\hat{\nu}}$, where $(\eta_{\mu\nu}) = \mathrm{diag}(\ 1,\ -1,\ -1,\ -1)$ is the Minkowski matrix.

The defining equations (2.1) show that in this paper the g_{zz} metric field component is constant, in contrast to [5, 7], where the case with constant g_{tt} component was studied.

Let the spacetime is filled with the charged perfect fluid and massless scalar field ϕ. The perfect fluid is characterized by its pressure p and energy density μ, from symmetry considerations both these quantities depending only on the radial coordinate. On the other hand ϕ may depend also on the longitudinal direction z. The electromagnetic field is non-minimally coupled to ϕ with parameter ε. Here we would like to point out that the spacetime with basis fields (2.1) is referred to cylindrically symmetric althought the dilaton generally depends also on the longitudinal direction.

The ε-order EMD action that we will deal with is given by

$$S_{\mathrm{eff}}[g, A, \phi, \mu] = \int\limits_{\mathcal{M}} \left[*R + 16\pi * \mu -\ \mathrm{d}\phi \wedge * \mathrm{d}\phi + 2\varepsilon\ \mathrm{e}^{\phi}\ F \wedge *F \right]\ , \tag{2.2}$$

where ε is a real parameter, R is the Ricci scalar of the metric tensor, F is the electromagnetic field 2-form. The scalar field ϕ will be henceforth called dilaton.

We claim to obtain a EMD solution which is of the first order in the parameter ε. Of course at the first place it means one should have a zero-order solution in ε that solves equations of motion for the action (2.2) if we let ε to be zero, or in other words, a purely classical solution of the Einstein equations coupled with a scalar field and perfect fluid. Then the ε-order solution we are looking for is naturally viewed as electromagnetic first order correction of the classical solution, and it has to coincide with the latter when ε goes to zero.

As for the dilaton ϕ, it may be written as the sum of zero-order solution $\phi^{(0)}$ and a ε-order correction like

$$\phi = \phi^{(0)} + \varepsilon\phi^{(1)}\ . \tag{2.3}$$

If only terms linear in ε are considered, one is forced to keep terms of ε-order in the corresponding equations of motion. Particularly, the coupling fuction

standing at $F \wedge *F$, being already of $\mathcal{O}(\varepsilon)$ becomes in our approximation equal to $2\varepsilon \mathrm{e}^{\phi^{(0)}}$.

An usual progress is to introduce a fluid comoving system, in which the fluid particles motion is uniquely determined by the velocity (co)vector field u, $u = \Theta^{\hat{0}}$. Let us very briefly mention basic properties of the geometry of fluid particles worldlines congruences. An acceleration 1-form $\dot{u}$ is given as $\dot{u} = -\,\mathrm{d}\alpha$. Since the problem is stationary, both expansion and shear tensor are vanishing. A vorticity covector is given by

$$\omega = \frac{1}{2} * (u \wedge \,\mathrm{d}u) = \frac{1}{2}\frac{\mathrm{d}f}{\mathrm{d}r} l^{-1} \mathrm{e}^{\alpha-\delta}\,\mathrm{d}z \ . \tag{2.4}$$

The last two statements show that the fermionic fluid rotates as a rigid body.

3 Zeroth-order solution

In this paragraph we derive a solution of zero-order in ε. In this case the action (2.2) becomes Einstein-dilaton plus fluid system. The Einstein field equations written in the tetrad representation (2.1) then read

$$-\frac{1}{2}\ \eta_{\alpha\beta\gamma} \wedge \Omega^{\hat{\beta}\hat{\gamma}} = 8\pi * i_{\hat{\alpha}} T \ , \tag{3.1}$$

where $i_{\hat{\alpha}} \equiv i_{\mathbf{e}_{\hat{\alpha}}}$ is the interior product ($\mathbf{e}_{\hat{\alpha}}$ is dual basis to (2.1), $\Theta^{\hat{\alpha}}(\mathbf{e}_{\hat{\beta}}) = \delta^{\alpha}_{\beta}$), Ω is curvature 2-form on $T\mathcal{M}$, $\Omega^{\hat{\alpha}}_{\hat{\beta}} = \frac{1}{2} R^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}\hat{\delta}} \Theta^{\hat{\gamma}} \wedge \Theta^{\hat{\delta}}$, and 1-form $\eta^{\alpha\beta\gamma}$ is defined by [8]

$$\eta^{\alpha\beta\gamma} = *(\Theta^{\hat{\alpha}} \wedge \Theta^{\hat{\beta}} \wedge \Theta^{\hat{\gamma}}) \ .$$

Finally T is the total stress-energy tensor of the perfect fluid and the masseless scalar field,

$$\begin{aligned} 8\pi T &= 8\pi \left[(\mu + p) u \otimes u - p\ g \right] \\ &+ \tfrac{1}{2} \left[\,\mathrm{d}\phi \otimes \,\mathrm{d}\phi - \tfrac{1}{2}\ g(\,\mathrm{d}\phi, \,\mathrm{d}\phi)\ g \right] \ . \end{aligned} \tag{3.2}$$

Explicit form of the Einstein equations takes the form

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\mathrm{e}^{2\alpha}}{M}\frac{\mathrm{d}f}{\mathrm{d}r}\right] = 0 \ , \tag{3.3a}$$

$$\frac{2}{M}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{M}\frac{\mathrm{d}\alpha}{\mathrm{d}r}\right] = -\frac{1}{M^2}\left(\frac{\partial\phi}{\partial r}\right)^2 , \tag{3.3b}$$

$$\frac{1}{M}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\mathrm{e}^{-2\alpha}}{M}\frac{\mathrm{d}}{\mathrm{d}r}\left(l^2 \mathrm{e}^{2\alpha}\right)\right] = 32\pi p\,\mathrm{e}^{2\alpha} \ , \tag{3.3c}$$

$$l^2 \frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial z} = 0 \ , \tag{3.3d}$$

$$\frac{\mathrm{e}^{-2\alpha}}{M}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{M}\frac{\mathrm{d}l^2}{\mathrm{d}r}\right] = 16\pi(p-\mu) + \frac{1}{M^2}\left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 - \left(\frac{\partial\phi}{\partial z}\right)^2 , \tag{3.3e}$$

$$2\mathrm{e}^{-2\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}r}\frac{\mathrm{d}\,l^2}{\mathrm{d}r} = 32\pi M^2 p - \left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 + \; l^2\left(\frac{\partial\phi}{\partial r}\right)^2 \mathrm{e}^{-2\alpha} - \left(\frac{\partial\phi}{\partial z}\right)^2 M^2 \;, \tag{3.3f}$$

Let us introduce the functions m and M by the formulae

$$M = l\mathrm{e}^{\delta-\alpha} \;, \qquad m = \int M\,\mathrm{d}r \;.$$

The scalar field equation of motion is the massless Klein-Gordon equation

$$*\;\; \mathrm{d}*\mathrm{d}\;\phi = \frac{1}{M}\frac{\partial}{\partial r}\left(\frac{l^2}{M}\frac{\partial\phi}{\partial r}\right) + \frac{\partial^2\phi}{\partial z^2} = 0\,. \tag{3.4}$$

The Bianchi identity, provided that the dilatonic equation of motion (3.4) is satisfied, becomes

$$u \wedge * \left[\mu\,\mathrm{d}u + \,\mathrm{d}(pu)\right] = 0 \;. \tag{3.5}$$

Because of the independence of r and z coordinates in (3.3) one can carry out the separation of variables in (3.4) to obtain ϕ in terms of the metric functions,

$$\phi = \phi_0 + \phi_1 z + \phi_2 \int \frac{M}{l^2}\;\mathrm{d}r\,, \quad \phi_1\phi_2 = 0 \;, \tag{3.6}$$

with constants ϕ_0, ϕ_1 and ϕ_2.

Thus one has reduced the problem to solving five equations (3.3a)-(3.3f) minus (3.3d) for six unknowns: α, f, l, δ and physical quantities of pressure p and mass (energy) density μ.

Inserting of (3.3a), (3.3b) and (3.3c) into (3.3f) yields system of two second-order equations for α and l^2 that reads

$$2l^4\frac{\mathrm{d}^2\alpha}{\mathrm{d}m^2} = -{\phi_2}^2 \;, \tag{3.7a}$$

$$\frac{\mathrm{d}^2 l^2}{\mathrm{d}m^2} = {\phi_1}^2\mathrm{e}^{2\alpha} + 4\Omega^2\mathrm{e}^{-2\alpha} \;, \tag{3.7b}$$

The authors have been able to find a solution to (3.7) if $\phi_2 = 0$, which from (2.3) and (3.6) immediately implies

$$\phi^{(0)} = \phi_0 + \phi_1 z \;. \tag{3.8}$$

This especially simple linear dependence of the dilaton is common in papers [2, 3, 4]. It also naturally emerges once one admits the dilatonic dependence only on the coordinate along the rotation axis. Since the dilaton blows up at the z-infinities, they can be condidered as additional sources of scalar charge.

The solution of the Einstein equations can be written in the form

$$\begin{aligned} \mathrm{d}s^2 \;\; &= \;\; \mathrm{e}^{2\alpha}\left(\,\mathrm{d}t + f\,\mathrm{d}\varphi\right)^2 - l^2\,\mathrm{d}\varphi^2 \\ &- \;\; \mathrm{d}z^2 - C^{-2}l^{-2}\left(\,\mathrm{d}\mathrm{e}^{\alpha}\right)^2 \;, \end{aligned} \tag{3.9}$$

the metric functions f and l^2 being given by

$$f = -\frac{\Omega}{C}\mathrm{e}^{-2\alpha} + F \ , \tag{3.10a}$$

$$C^2 l^2 = \Omega^2 \mathrm{e}^{-2\alpha} + \frac{1}{4}\phi_1^2 \mathrm{e}^{2\alpha} + D\alpha + E \ , \tag{3.10b}$$

with $\Omega,\ C,\ D,\ E,\ F$ integration constants. The physical quantities, the energy density and the pressure, are found to be

$$\begin{aligned} 16\pi\mu &= D\mathrm{e}^{-2\alpha} - \phi_1^2 \ , \\ 16\pi p &= D\mathrm{e}^{-2\alpha} + \phi_1^2 \ . \end{aligned} \tag{3.11}$$

The formulae (3.10) and (3.11) are expressed in terms of an arbitrary non-constant C^2 function α that reflects the radial coordinate rescaling possibility.

4 First-order solution

An electromagnetic field is represented by a 2-form F in the action (2.2). The electromagnetic field, being already of ε-order, is non-minimally coupled to gravity with the firm (exponential) dependence on the longitudinal direction. Our next task is to take a suitable *Ansatz* for the electromagnetic field and then solve the equations of motion.

Let charge be distributed with a current density $j(r, z)$ through a spacetime. Note we have not included the source term $A \wedge {*j}$, where A is a vector potential, into the action (2.2) because of technical simplicity. This is possible if and only if $A \wedge {*j}$ is an exact form and can be transformed away.

Of great physical importance, in particular on the field of rotating spacetimes we deal with, is the case when the Lorentz force, in the comoving system proportional to $*(u \wedge {*F})$, acting on the fluid particles, vanishes. In the fluid rest frame it means that only a magnetic field survives. The form of the metric field equations of motion, namely the φz and φr components, leads us to exclude the spacetime with electric currents parallel to the axis of rotation, in which the angular part of the magnetic field vanishes identically. The electromagnetic field 2-form is then given by

$$F = \mathrm{B}_{\hat{r}}\ \Theta^{\hat{1}} \wedge \Theta^{\hat{2}} + \mathrm{B}_{\hat{z}}\ \Theta^{\hat{3}} \wedge \Theta^{\hat{1}} \ . \tag{4.1}$$

The presence of the radial magnetic field may seem to be artificial because it causes a strange phenomena - an occurence of magnetic charges (monopoles). In fact, this is the case. But the form of zr-component of the Einstein equations, namely the equation (4.2d), enforces the existence of the radially pointing magnetic field in order for the dilaton to be also radially dependent. Otherwise it would simply be given by (3.8).

The metric field equations of motion following from the action (2.2) are the Einstein equations (3.1) with the stress-energy tensor (3.2) enriched by

the electromagnetic field contribution [8], where electromagnetic field is non-minimally coupled to gravity,

$$T_{\rm elmag} = \varepsilon \frac{\mathrm{e}^{\phi^{(0)}}}{8\pi}\ \Theta^{\hat\alpha} \otimes *(F \wedge i_{\hat\alpha} * F - i_{\hat\alpha}\ F \wedge *F)\ .$$

The appropriate Einstein-Maxwell system reads

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\mathrm{e}^{2\alpha}}{M}\frac{\mathrm{d}f}{\mathrm{d}r}\right] = 0\ , \tag{4.2a}$$

$$\frac{2}{M}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{M}\frac{\mathrm{d}\alpha}{\mathrm{d}r}\right] + \frac{1}{M^2}\left(\frac{\partial\phi}{\partial r}\right)^2 = \frac{4\varepsilon}{l^4}\mathrm{e}^{\phi+2\alpha}F_{\varphi z}^2\ , \tag{4.2b}$$

$$\frac{1}{M}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\mathrm{e}^{-2\alpha}}{M}\frac{\mathrm{d}}{\mathrm{d}r}\left(l^2\mathrm{e}^{2\alpha}\right)\right] = 32\pi p\,\mathrm{e}^{2\alpha}\ , \tag{4.2c}$$

$$l^2\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial z} = 4\varepsilon\mathrm{e}^{\phi}F_{r\varphi}F_{\varphi z}\ , \tag{4.2d}$$

$$\frac{\mathrm{e}^{-2\alpha}}{M}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{M}\frac{\mathrm{d}l^2}{\mathrm{d}r}\right] = 16\pi(p-\mu) + \frac{1}{M^2}\left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 - \left(\frac{\partial\phi}{\partial z}\right)^2 - 4\varepsilon\mathrm{e}^{\phi}\frac{F_{\varphi z}^2}{l^2}, \tag{4.2e}$$

$$\begin{aligned} 2\mathrm{e}^{-2\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}r}\frac{\mathrm{d}\,l^2}{\mathrm{d}r} &= 32\pi M^2 p - \left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 + 4\varepsilon\mathrm{e}^{\phi-2\alpha}F_{r\varphi}^2 - 4\varepsilon\mathrm{e}^{\phi}\frac{M^2}{l^2}F_{\varphi z}^2 \\ &+ l^2\left(\frac{\partial\phi}{\partial r}\right)^2\mathrm{e}^{-2\alpha} - \left(\frac{\partial\phi}{\partial z}\right)^2 M^2\ , \end{aligned} \tag{4.2f}$$

and it is to be completed by the massless Klein-Gordon equation (3.6), which does not undergo any changes, and furthermore by the modified Maxwell equations. In (4.2) as well as in the remainder of this section e^{ϕ} stands for $\mathrm{e}^{\phi^{(0)}}$.

Variation of the action (2.2) with respect to a vector potential A yields the generalized Maxwell equations

$$-*\mathrm{d}*(\mathrm{e}^{\phi^{(0)}}\ F) = \frac{4\pi}{\varepsilon}\ j\ , \tag{4.3}$$

or in the explicit form

$$\begin{aligned} &l^2\mathrm{e}^{\phi-2\alpha}\frac{\partial}{\partial r}\left[(f\delta_0^\mu - \delta_1^\mu)\frac{F_{r\varphi}}{M}\right] \\ &+M(f\delta_0^\mu - \delta_1^\mu)\frac{\partial}{\partial z}(\mathrm{e}^{\phi}F_{z\varphi}) = -\frac{4\pi}{\varepsilon}Ml^2j^\mu\ . \end{aligned} \tag{4.4}$$

The same procedure as in the zero-order case gives the following system for functions l^2 and α

$$2l^4\frac{\mathrm{d}^2\alpha}{\mathrm{d}m^2} = 4\varepsilon\mathrm{e}^{\phi+2\alpha}F_{\varphi z}^2 - \tilde\phi_2^2\ , \tag{4.5a}$$

$$\frac{\mathrm{d}^2l^2}{\mathrm{d}m^2} = \tilde\phi_1^2\mathrm{e}^{2\alpha} + 4\Omega^2\mathrm{e}^{-2\alpha} - 4\varepsilon\mathrm{e}^{\phi}\frac{F_{r\varphi}^2}{M^2} \tag{4.5b}$$

with new constants $\tilde{\phi}_1$ and $\tilde{\phi}_2$ which already include the ε-order correction.

Because the dilaton is written as (2.3) and $\phi^{(0)}$ does not depend on r, the term $(\frac{\partial\phi}{\partial r})^2$ (and also $\tilde{\phi}_2^2$) is already of $\mathcal{O}(\varepsilon^2)$ and should be neglected. Putting together equations (4.2d), (4.5a) and (4.4) we have arrived at the following conditions for the electromagnetic field

$$\begin{aligned} *(u\wedge F)\wedge\Theta^{\hat{2}} = 0 \ , \\ u\wedge *F = 0 \ . \end{aligned} \tag{4.6}$$

The equations (4.2) and (4.6) are solved by a purely longitudinal magnetic field, parallel to the rotation axis

$$\mathrm{B}_{\hat{z}} = B\mathrm{e}^{-\frac{1}{2}\phi^{(0)}-\alpha} \ , \qquad B = \mathrm{const} \tag{4.7}$$

while the radially pointing magnetic field vanishes, i.e. $\mathrm{B}_{\hat{r}} = 0$ in (4.1). As a matter of fact one has quite transparent physical interpretation of the found result. Since according to (4.7) and (4.3) it must hold $j\wedge\Theta^{\hat{0}} = 0$, we conclude that the fluid particles are the charge carriers, i.e. the current density is purely convectional, $j = \rho u$. The charge density ρ is determined by the formula

$$4\pi\rho = -\varepsilon\frac{B}{M}\frac{\mathrm{d}f}{\mathrm{d}r}\mathrm{e}^{\frac{1}{2}\phi^{(0)}-\alpha} \ . \tag{4.8}$$

But there is a price we must pay for the simplification. Note that the exterior derivative of (4.7) does not vanish which means that we deal with a current of the magnetic monopoles and one has to introduce a magnetic charges current density 1-form j_m by

$$\frac{4\pi}{\varepsilon}\, j_m = *\,\mathrm{d}F = -\frac{1}{2}B\phi_1\mathrm{e}^{-\frac{1}{2}\phi^{(0)}-\alpha}\Theta^{\hat{0}} \ . \tag{4.9}$$

Equations (4.8) and (4.9) show us that the source term $A\wedge *j$ is identically vanishing.

Essentially there are two possibilities to keep this situation physically acceptable. Either we can expect that going to a non-abelian gauge fields will smooth out this solution, or, in the case of abelian gauge fields, it is possible to introduce j_m explicitly in the action (2.2), but one has to break the general covariance to do this [9].

Now we can straightforwardly solve the Einstein equations. From the same reason as in the zero-order case one has one degree of freedom corresponding to the radial coordinate rescalling possibility. One finds that one has seven independent equations for exactly eight unknowns: four metric functions $f,\ l,\ \alpha,\ \delta$ and four physical quantities $p,\ \phi,\ \mu,\ \rho$.

The dilaton according to the (3.6) becomes equal to

$$\phi = \phi_0 + \tilde{\phi}_1 z \ .$$

The Bianchi identity in our case, provided that the scalar field equation of motion (3.6) is fulfilled, is

$$u\wedge *(\mu\ \mathrm{d}u + \mathrm{d}(p\ u) + \varepsilon\ \rho\ \mathrm{e}^{\phi^{(0)}}\ F) = 0 \ . \tag{4.10}$$

After all one obtains the result (3.9) with the following functions f and l

$$\begin{aligned} f &= -\frac{\Omega}{C}\mathrm{e}^{-2\alpha} + F \ , \\ C^2 l^2 &= \Omega^2 \mathrm{e}^{-2\alpha} + \frac{1}{4}\tilde{\phi}_1^2 \mathrm{e}^{2\alpha} - 2\varepsilon B^2 \alpha^2 + D\alpha + E. \end{aligned} \tag{4.11}$$

For the energy density, the pressure and the charge density (4.8) one has

$$\begin{aligned} 16\pi\mu &= \left[D + 2\varepsilon B^2 \left(1 - 2\alpha\right)\right] \mathrm{e}^{-2\alpha} - \tilde{\phi}_1^2 \ , \\ 16\pi p &= \left[D - 2\varepsilon B^2 \left(1 + 2\alpha\right)\right] \mathrm{e}^{-2\alpha} + \tilde{\phi}_1^2 \ , \\ 2\pi\rho &= -\varepsilon\Omega B \mathrm{e}^{\frac{1}{2}\phi^{(0)} - 3\alpha} \ . \end{aligned} \tag{4.12}$$

The mathematical structure of the solution (4.11) and (4.12) of the Einstein-Maxwell equations (4.2) is much the same as the zeroth-order one (3.10) and (3.11). The differences appear in the presence of the term quadratic in α in the function l^2 in (4.11) and the linear terms in α in (4.12), which occurs due to the existence of the magnetic field. The function f remains unchanged.

5 Case with more scalar fields

It is straightforward to generalize the zeroth order solution (3.10) and the first order solution (4.11) in the case when more scalar fields are present. The motivation comes from string theory, where it is known that the effective description at low energies may contain not only the dilaton, but also others tensor fields, depending on how the compactification was carried out [10]. Among them is most important an axionic tensor field that can be, just in four dimensions, represented by an extra massless scalar field. Also some additional massless scalar fields, called modulus fields, may be present [3].

We shall consider N massless scalar fields ϕ_i, $i = 1, 2, ..N$, and N non-minimally coupled massless scalar fields ψ_i. The total action (2.2) can be rewritten as

$$\begin{aligned} S_{\text{eff}} = \int\limits_{\mathcal{M}} & \left[*R + 16\pi * \mu + 2\varepsilon \ \mathrm{e}^{\phi} \ F \wedge *F \right. \\ & \left. - \sum_i \left(\mathrm{d}\phi_i \wedge * \mathrm{d}\phi_i + \mathrm{e}^{-2\phi_i} \ \mathrm{d}\psi_i \wedge * \mathrm{d}\psi_i \right) \right] \ . \end{aligned} \tag{5.1}$$

Before proceeding further let us mention that each scalar field ϕ_i or ψ_i can be written a similar way to equation (2.3).

5.1 Zeroth order in ε

The Klein-Gordon equation (3.4) for each of the scalar fields ϕ_i still holds, while the equations of motion for the scalar fields ψ_i are given by

$$* \ \mathrm{d}(\mathrm{e}^{-2\phi_i} * \mathrm{d}\psi_i) = 0 \ , \tag{5.2}$$

for each index i. The modified Einstein's field equation are listed below. Again, as in section 3, the authors were able to solve the generalization of (3.7) provided that neither ϕ_i nor ψ_i depends on the radial coordinate. Then from (5.6) we have

$$\phi_i = \phi_{i0} + \phi_{i1} z \ , \ \psi_i = \psi_{i0} + \mathrm{e}^{\phi_i} \psi_{i1} \ , \tag{5.3}$$

which inserted into the (5.2) yields $\phi_{i1}\psi_{i1} = 0$. In (5.3) $\phi_{i0}, \phi_{i1}, \psi_{i0}, \psi_{i1}$ are integration constants. Thus the only non-trivial zero-order solution is given by

$$\phi_i^{(0)} = \phi_{i0} + \phi_{i1} z \ , \ \psi_i^{(0)} = \psi_{i0} \ . \tag{5.4}$$

The solution (3.10) and (3.11) remains unaffected provided ϕ_1^2 is replaced by $\Phi_1^2 = \sum \phi_{i1}^2$.

5.2 First order in ε

As a consequence of the presence of more massless scalar fields, one has to modify the equation (4.2) in the following manner. The equation (4.2b) becomes

$$\begin{aligned} &\frac{2}{M} \frac{\mathrm{d}}{\mathrm{d}r} \left[\frac{1}{M} \frac{\mathrm{d}\alpha}{\mathrm{d}r} \right] - \frac{4\varepsilon}{l^4} \mathrm{e}^{\phi + 2\alpha} F_{\varphi z}^2 \\ &= -\frac{1}{M^2} \sum_i \left[\left(\frac{\partial \phi_i}{\partial r} \right)^2 + \mathrm{e}^{-2\phi_i} \left(\frac{\partial \psi_i}{\partial r} \right)^2 \right] \ . \end{aligned} \tag{5.5}$$

The term $(\frac{\partial \phi}{\partial z})^2$ on the right-hand side of the equation (4.2e) should be replaced by

$$\sum_i \left[\left(\frac{\partial \phi_i}{\partial z} \right)^2 + \mathrm{e}^{-2\phi_i} \left(\frac{\partial \psi_i}{\partial z} \right)^2 \right] \ . \tag{5.6}$$

Similarly the equation (4.2f) will be changed in an obvious way. Finally (4.2d) becomes

$$l^2 \sum_i \left(\frac{\partial \phi_i}{\partial r} \frac{\partial \phi_i}{\partial z} + \mathrm{e}^{-2\phi_i} \frac{\partial \psi_i}{\partial r} \frac{\partial \psi_i}{\partial z} \right) = 4\varepsilon \mathrm{e}^{\phi} F_{r\varphi} F_{\varphi z} \ . \tag{5.7}$$

It was stated before that the scalar fields are decomposed into zero-order part and first-order correction as

$$\phi_i = \phi_{i0} + \phi_{i1} z + \varepsilon \phi_i^{(1)} \ , \ \psi_i = \psi_{i0} + \varepsilon \psi_i^{(1)} \ .$$

As a matter of fact it is seen that all terms in the modified Einstein's equations involving $(\frac{\partial \psi_i}{\partial r})^2, (\frac{\partial \psi_i}{\partial z})^2$ and even $(\frac{\partial \phi_i}{\partial r})^2$ are already of $\mathcal{O}(\varepsilon^2)$ and have to be ignored. Therefore we continue to keep our *Ansatz* (4.7) for the electromagnetic field. From this fact it immediately follows that the term $F \wedge F$ vanishes identically. The equation (4.9) should be modified due to the presence of the zero-order axionic field. But $\psi^{(0)} = \psi_0$ is constant, which can be set equal to one, and so in particular the equation (4.9) applies in this case as well. It also means that, for example, from (4.2d), the scalar fields ϕ_i are given by

$$\phi_i = \phi_{i0} + \tilde{\phi}_{i1} z \ , \tag{5.8}$$

with constants $\tilde{\phi}_{i1}$ including the ε-order corrections to ϕ_{i1}. The resulting metric and physical quantities are still given by (4.11) and (4.12) provided $\tilde{\phi}_1^2$ is replaced by $\tilde{\Phi}_1^2 = \sum \tilde{\phi}_{i1}^2$.

Since the Einstein's equations give us no information with respect to the fields ψ_i one has to solve their equations of motion (5.2). One can carry out the separation of variables to obtain

$$\psi_i = \psi_{i0} + \varepsilon \mathrm{e}^{\phi_{i1} z} [A_i \cos(v_i z) + B_i \sin(v_i z)] \; \eta_i(r) \; , \tag{5.9}$$

where $A_i,\ B_i$ and v_i are arbitrary constants and the functions η_i are solutions of second-order equations that can be transformed into the form

$$\mathrm{e}^{-2\alpha} \frac{\mathrm{d}}{\mathrm{d}m} \left(l^2 \frac{\mathrm{d}\eta_i}{\mathrm{d}m} \right) - (\phi_{i1}^2 + v_i^2) \; \eta_i = 0 \; . \tag{5.10}$$

6 String-inspired theory of gravity

The aim of this section is to show that the solution described by the metric (4.11) actually remains unaltered even if string-inspired charged gravity is taken under consideration [3].

The total string-inspired effective action (2.2) can be rewritten as [3, 11]

$$\begin{aligned} S_{\mathrm{eff}} = \int\limits_{\mathcal{M}} & [*R + 16\pi * \mu \\ & - \sum_i (\,\mathrm{d}\phi_i \wedge * \,\mathrm{d}\phi_i + \mathrm{e}^{-2\phi_i} \,\mathrm{d}\psi_i \wedge * \,\mathrm{d}\psi_i) \\ & - \, 8\pi^2 \varepsilon \mathrm{e}^{\phi} e(\mathcal{M}) + 4\varepsilon \; \psi \; \mathrm{Tr} \; \Omega \wedge \Omega \\ & + \, 2\varepsilon \; \mathrm{e}^{\phi} \; F \wedge *F - 4\varepsilon \; \psi \; F \wedge F \big] \; . \end{aligned} \tag{6.1}$$

In terms of the inverse string tension α' (Regge slope) and the string coupling constant g the parameter ε is expressed like $\varepsilon = \frac{\alpha'}{4\mathrm{g}^2}$. The Euler class $e(\mathcal{M})$ of $T\mathcal{M}$ occuring in (6.1) is in four dimensions equal to [10]

$$e(\mathcal{M}) = \frac{1}{8\pi^2} \left(R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - 4 R_{\alpha\beta} R^{\alpha\beta} + R^2 \right) \eta \; , \tag{6.2}$$

η is the volume element with components $\eta_{\alpha\beta\gamma\delta}$.

We have also added an extra contribution arising from a field μ. Physically it may represent an energy density of a fermionic matter, that is in our model approximated by a perfect fluid of a pressure p. Although this picture is rather intuitive and is not as transparent as in former EMD, later it will turn out to be useful. If one wants to supress the fermionic matter and recover an ordinary string-inspired action with the cosmological constant Λ, then our approach is fruitful too since the state equation $\mu + p = 0$ along with $\Lambda = -8\pi p$ gives the desired modification of the action. The fields $\phi_N \equiv \phi$ and $\psi_N \equiv \psi$ may be referred to as the dilaton and axion respectively.

For any cylindrically symmetric stationary metric, i.e. metric depending on the radial coordinate with the only non-vanishing cross-term $g_{\varphi t}$, a straightforward calculation gives the following useful formula for the Euler class (6.2)

$$\begin{aligned} e(\mathcal{M}) &= \frac{1}{4\pi^2}\frac{\mathrm{d}}{\mathrm{d}r}\left\{\left[\left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 e^{2\alpha}+2\frac{\mathrm{d}\alpha}{\mathrm{d}r}\frac{\mathrm{d}l^2}{\mathrm{d}r}\right] l^{-1}\mathrm{e}^{\alpha+\gamma-3\delta}\frac{\mathrm{d}\gamma}{\mathrm{d}r}\right\} \\ &\times \quad l^{-1}\mathrm{e}^{-\alpha-\gamma-\delta}\ \Theta^{\hat{0}}\wedge\Theta^{\hat{1}}\wedge\Theta^{\hat{2}}\wedge\Theta^{\hat{3}}\ . \end{aligned} \tag{6.3}$$

In (6.3) the function γ is given by $g_{zz} = -\mathrm{e}^{2\gamma}$. Thus for (3.9) the Euler class vanishes. For the basis (2.1) it turns out that the term

$$\mathrm{Tr}\ \Omega\wedge\Omega = -\frac{1}{4}R^{\alpha\beta}{}_{\gamma\delta}R^{\sigma\tau}{}_{\alpha\beta}\ \eta^{\gamma\delta}{}_{\sigma\tau}\ \eta$$

is identically vanishing.

Since the magnetic field (4.7) is purely longitudinal, the term $F\wedge F$ is vanishing too and our problem in fact reduces to the Einstein-Maxwell-dilaton plus fluid system disscussed in previous sections. This completes the proof that the metric (4.11) after appropriate replacement $\tilde{\phi}_1^2 \to \tilde{\Phi}_1^2 = \sum \tilde{\phi}_{i1}^2$ constitutes string-inspired solution. The scalar fields are given by (5.8) and (5.9), subject to the equation (5.10).

7 On some atributes of the solution

We briefly comment on some physical atributes of the solution (4.11).

Of course no every specialization of the integration constants in (4.11) leads to cylindrically symmetric spacetime. If one requires the found solutions to be cylindrically symmetric and regular at the origin the axial symmetry condition and the elementary flatness condition have to be imposed [12]. In our case, provided $\alpha \propto r^2$ for small vaues of r, these conditions give

$$\begin{aligned} \Omega^2 - \frac{1}{4}\tilde{\phi}_1^2 - \frac{D}{2} &= \pm\ C\ , \\ \Omega^2 + \frac{1}{4}\tilde{\phi}_1^2 + E &= 0\ , \\ F &= \frac{\Omega}{C}\ . \end{aligned} \tag{7.1}$$

Clearly the Lorentz force is vanishing. Also it can straightforwardly seen that the source term $A\wedge *j$ is exact form. Furthermore non-geodesic motion of the fluid should be understood as a mere consequence of a pressure inhomogeneity. Indeed it follows from fact that the accelaration is $\dot{u} = -\,\mathrm{d}\alpha$ and that the Bianchi identity, (3.5) or (4.10), can be rewritten as

$$(\mu+p)\,\frac{\mathrm{d}\alpha}{\mathrm{d}r} = -\frac{\mathrm{d}p}{\mathrm{d}r}\ .$$

The vorticity 1-form ω according to (2.4) equals

$$\omega = \frac{1}{2} * (u\wedge\,\mathrm{d}u) = \Omega\mathrm{e}^{-2\alpha}\,\mathrm{d}z\ .$$

In this way the metric (4.11) is static if and only if $\Omega = 0$.

Let us also write down how the energy conditions restrict the ranges of the integration constants in (4.11). The strong along with the dominant energy condition imply the following two inequalities

$$
\begin{aligned}
D - 4\varepsilon B^2\alpha + \frac{1}{2}\tilde{\phi}_1^2 \mathrm{e}^{2\alpha} &\geq 0 \ , \\
4\varepsilon B^2 &\geq \tilde{\phi}_1^2 \mathrm{e}^{2\alpha} \ .
\end{aligned}
\tag{7.2}
$$

Next remark concernes the algebraic classification of the Weyl tensor. It turns out that the metric (4.11) is of Petrov type D except on the hypersurfaces (one or more) given by

$$
4\varepsilon B^2\alpha = 2\varepsilon B^2 + D \ ,
$$

where it is of type O.

Last remark clarifies the connection between (4.11) and Gödel-type solutions [4], provided that the function α and integration constants in (4.11) are chosen conviniently. The following appropriate choice respecting regularity conditions (7.1) has been done

$$
\begin{aligned}
&\Omega^2 = \varepsilon B^2 + \frac{1}{2a^2} - \frac{1}{4}\tilde{\phi}_1^2 \ , \ E = -\left(\frac{1}{2a^2} + \varepsilon B^2\right) \ , \\
&D = \frac{1}{a^2} + 2\varepsilon B^2 - \tilde{\phi}_1^2 + 2C \ , \ F = \frac{\Omega}{C} \ .
\end{aligned}
\tag{7.3}
$$

The physical meaning of the constant a will be clear shortly. Let us now specify the arbitrary function α as $\alpha = 2a^2 C \mathrm{sh}^2(\frac{r}{2a})$. For simplicity let us consider only the dilatonic and axionic fields. We obtain new metric that depends upon three parameters: a, B and C (explicit form is omitted here). This solution describes an inhomogeneous universe and from (7.2) it is immediate that C must not be positive. If C is set equal to zero, the general solution (4.11) becomes

$$
\mathrm{d}s^2 = [\mathrm{d}t + 4a^2\Omega \mathrm{sh}^2(\frac{r}{2a})\,\mathrm{d}\varphi]^2 - a^2\mathrm{sh}^2(\frac{r}{a})\,\mathrm{d}\varphi^2 - \mathrm{d}z^2 - \mathrm{d}r^2, \tag{7.4}
$$

which is manifestly of the Gödel-type. If in addition the state equation $p + \mu = 0$ is requested, the scalar field contribution must have the form

$$
\tilde{\phi}_1^2 = \frac{1}{a^2} + 2\varepsilon B^2 \ . \tag{7.5}
$$

The latter equation reflects fact that the non-vanishing cosmological constant rather then the perfect fluid is considered. If we let $\varepsilon \to 0$ then from the section 3, especially from the equation (3.8), one has $a^2\phi_1^2 = 1$, and from (2.3), (5.8) and (7.5) the ε-order corrected dilaton is given by

$$
\phi = \phi_0 + \phi_1 z \left(1 + \varepsilon B^2 a\right) \ . \tag{7.6}
$$

The relationship between fundamental parameters of the theory becomes

$$
4\Omega^2 - \frac{1}{a^2} = 2\varepsilon B^2 \ , \tag{7.7}
$$

subject to the inequality $2\varepsilon a^2 B^2 \geq 1$, which follows from the energy conditions (7.2).

For the axion one has the equations (5.9) and (5.10). With help of the elementary theory of Legendre polynomials we find $v = \phi_1$ and

$$\psi = \psi_0 + \varepsilon \mathrm{e}^{\phi_1 z} \mathrm{ch}\left(\frac{r}{a}\right)\left[A\cos\left(\frac{z}{a}\right) + B\sin\left(\frac{z}{a}\right)\right] .$$

Note that in zero-order regime $\varepsilon \to 0$ one has $a^2\phi_1^2 = 1$ and $4\Omega^2 = a^{-2}$, the latter equality immediately implying $g_{\varphi\varphi} \leq 0$. Thus there are no CTC's in the spacetime. On the other hand in ε-order framework, since because of (7.2) is ε generally non-negative, $g_{\varphi\varphi}$ becomes positive for sufficiently large r. Thus the first order corrections cause the chronology violation.

Equations (7.4), (7.6) and (7.7) are identical with these of Kanti and Vayonakis [3] in string-inspired charged gravity framework, when $\varepsilon = \frac{\alpha'}{4\mathrm{g}^2}$. Their α'-order solution arising from Som and Raychaudhuri *Ansatz* for the electromagnetic field turned out to be most favored between others cases, also belonging to the α'-order.

We could also obtain proper generalization of another solution in [3] with a positive cosmological constant, simply by carrying out the imaginary transformation $a \to ia$ in (7.3), but we will not follow this line further.

8 Discussion

In the paper a class of stationary symmetric spacetimes within the framework of Einstein-Maxwell-dilaton gravity was found, that exhibits cylindrical symmetry. This solution is exact up to first order in parameter ε. Provided that the scalar fields in zeroth order do not depend on the radial coordinate we were also able to find the generalization to the case when more scalar fields are present.

The solution obtained depends upon the dilaton gradient and the magnetic field in the longitudinal direction. It is worthwhile to note that the Gauss-Bonnet term vanishes. Namely for this reason is found metric exact α'-order solution in string-inspired theory framework.

Since the forms of zero and first order solutions are similar, we can straightforwardly find out what is a consequence of the electromagnetic field presence with respect to the chronology violation. It turns out that it has the power to break down the chronology even if the zero-order solution was chronologically well behaved.

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